Massless Particles

The non-relativistic equations for kinetic energy and momentum both depend on the mass of the object and its velocity:

$$K = \tfrac{1}{2}v^2 \qquad\qquad p = mv$$

Notice how we could easily write the kinetic energy in terms of the momentum:

$$K = \frac{p^2}{2m}$$

Though we never used this equation, it is occasionally useful. One might wonder: is there a similar equation that relates momentum and energy for relativistic situations? There is!

$$E^2 = (pc)^2 + (mc^2)^2$$

This is not so obvious, so let's do the derivation below, but all it really is some algebra. Let's start with the equations for energy and momentum:

$$E = \gamma mc^2 \qquad\qquad p = \gamma mv$$

First, we will multiply the momentum by c, and then square both of the equations:

$$E^2 = \gamma^2 m^2 c^4 \qquad\qquad p^2c^2 = \gamma^2 m^2 v^2 c^2$$

Now subtract the momentum term from the energy and then do some algebra:

$$E^2 - p^2c^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 c^2$$

$$E^2 - p^2c^2 = \gamma^2 m^2 c^4 (1 - \frac{v^2}{c^2})$$

$$E^2 - p^2c^2 = \gamma^2 m^2 c^4 (\frac{1}{\gamma^2})$$

$$E^2 - p^2c^2 = m^2c^4$$

$$E^2 = (pc)^2 + (mc^2)^2$$

So why is this equation useful? It actually shows something very interesting – a particle could potentially have 0 mass, yet have energy and momentum! Notice how if m = 0, then

$$E = pc$$

Let's substitute in the actual equations for E and p to see what happens:

$$\gamma mc^2 = \gamma mvc$$

$$v = c$$

This means that a particle with a mass of 0 would have a speed of c! That is exactly what light is. A photon is a particle with 0 mass, travels with a speed of c, and has momentum and energy!